Visualizing Groups
What quintic polynomials have to do with Rubik’s Cube

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Puzzle #1: Rubik’s Cube

- **Erno Rubik, Budapest, Hungary, 1974**

- **How it works**
  - There is a “home” state
  - There are a few legal moves, subject to certain constraints
  - Moves can be repeated/undone
  - Moves are predictable

- **Goal: Return cube to home state**
The Rubik’s Cube Book

- Each page contains
  - A page number
  - How many steps it is from the home state
  - The cube’s appearance, both front and back
  - Where each move would lead
    - A page number
    - Closer or farther from home?

- A slow but sure way to solve the cube
  (43,252,003,274,489,856,000 pages)
Puzzle #2: The Rectangle

- Nathan Carter, Bloomington, IN, 2004
- How it works
  - There is a “home” state
  - There are a few legal moves, subject to certain constraints
  - Moves can be repeated/undone
  - Moves are predictable
- Goal: Return rectangle to home state
The Rectangle Book

- The Rectangle puzzle has few enough possible configurations that its “book” would be very small.
- In fact, we can put all the information about the rectangle in one small diagram.
The puzzles had these things in common

- Recognizable solved state
- Small set of legal moves
- Ability to repeat/undo moves
- Predictability of moves

These properties define what mathematicians call a *group*. 
They show up in the diagram these ways

- Mark one state as “home”
- Small set of arrow types
- Follow arrows sequentially/backwards
- Arrows are unambiguous

Diagrams like this are a good way to visualize groups.
Visualizing a Group

Rectangle Puzzle Map

Cayley Diagram

Klein 4-group, $V_4$
Some example groups shown via their Cayley diagrams

\[ \mathbb{Z}_3 \]  
\[ S_3 \]  
\[ Q_8 \]
Evariste Galois

1811-1832
Galois studied **polynomial equations** with **integer coefficients**: 

- **Quadratic**
  - **Quadratic formula**
    \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
    \[ ax^2 + bx + c = 0 \]

- **Cubic**
  - **Cardano’s formula**
    \[ ax^3 + bx^2 + cx + d = 0 \]

- **Quartic**
  - **Ferrari’s method**
    \[ ax^4 + bx^3 + cx^2 + dx + e = 0 \]

- **Quintic**
  - **No solution**
    \[ ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \]
Major Step #1: Galois Groups

The symmetries of the roots of a polynomial form a group.

- Recognizable home state
- Small set of legal moves subject to certain constraints
- Moves can be repeated/undone
- Moves are predictable
- Roots of a polynomial in any chosen order
- Reordering the roots subject to algebraic equations
- Reordering the roots
- Reorderings are predictable
Example Galois group

Four roots

\[
\begin{align*}
a &= \sqrt{2} + \sqrt{3} \\
b &= \sqrt{2} - \sqrt{3} \\
c &= -\sqrt{2} + \sqrt{3} \\
d &= -\sqrt{2} - \sqrt{3}
\end{align*}
\]

Polynomial

\[
x^4 - 10x + 1 = 0
\]

Equations relating the roots:

\[
\begin{align*}
a + d &= 0 \\
b + c &= 0 \\
(a + b)^2 &= 8 \\
(c + d)^2 &= 8 \\
(a + c)^2 &= 12 \\
(b + d)^2 &= 12
\end{align*}
\]
Group Theory Today

Frieze patterns (7 groups)

Wallpaper patterns (17 groups)

Chemical crystallography (230 groups)
Group Theory Classes

- Group theory is a standard part of undergraduate programs in pure mathematics
- Common teaching method: theorem-proof lectures, with examples
- Symmetry groups appear, but their uses are limited
- Cayley diagrams are rarely seen
Group Explorer

- Diagrams are helpful, but tedious to make by hand.
- Software would be very helpful in this area.
- Group Explorer original design team:
  - Brad Emmons
  - Douglas Hofstadter
  - Nathan Carter
- Version 1.5.8 for Windows XP since 2003
- Version 2.0 for Windows, Mac, and Unix since 2005
  - Users in at least 8 US states and at least 8 countries
  - Over 1000 downloads since December
Major Step #2: Main Theorem

One can tell which polynomials have solutions expressible by radicals by its Galois group.

Polynomial

\[ x^4 - 10x + 1 = 0 \]

\[ a = \sqrt{2} + \sqrt{3} \]
\[ b = \sqrt{2} - \sqrt{3} \]
\[ c = -\sqrt{2} + \sqrt{3} \]
\[ d = -\sqrt{2} - \sqrt{3} \]

Galois group

Solvable?

One can tell which polynomials have solutions expressible by radicals by its Galois group.

\[ 1 \triangleleft H_n \triangleleft \cdots \triangleleft H_2 \triangleleft H_1 \triangleleft G \]
Any quintic polynomial with three real and two complex roots will generate a group of roots that contains the group $A_5$, which has no normal subgroups.
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